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On the distribution of the number of monotone Boolean functions relative to the number of lower units

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Dedicated to Daniel Kleitman on his 65th birthday

Abstract

We find asymptotic formulae for the number of monotone Boolean functions of n variables with a most probable number of terms in the minimal disjunctive normal form. It is proven that the distribution of such functions is asymptotically normal if all monotone Boolean functions are equiprobable. The formulae are different depending on whether n is even or odd.

Keywords: Monotone Boolean function; Lower unit; Disjunctive normal form; Asymptotic

1. Introduction

The class of monotone Boolean functions is one of the most widely used and intensely studied classes of Boolean functions. Attention has been focused on it in such

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diverse fields as game theory [16], computational learning theory [14,13], harmonic analysis [2], and signal processing [19,17].

The number of monotone Boolean functions of n variables, denoted by $\psi(n)$, has interested researchers ever since Dedekind first posed the problem in 1897 [4]. While the value of $\psi(n)$ is known only for $n \leqslant 8$ [20], much work has gone into computing lower and upper bounds for $\psi(n)$, most notably by Kleitman [10,11] who showed that

$$\psi(n) \leqslant 2^{(1+O(\log n/n))\binom{n}{\lfloor n/2 \rfloor}}.$$

Although arbitrary Boolean functions can generally have multiple minimal disjunctive normal form (DNF) representations, monotone Boolean functions are uniquely represented by their minimal DNF. Furthermore, it is well known that a monotone Boolean function of n variables can have no more than $\binom{n}{\lfloor n/2 \rfloor}$ terms or, equivalently, conjunctions in its minimal DNF [7]. The number of terms in the minimal DNF of a monotone Boolean function is an important parameter. For example, in [8], monotone Boolean functions were studied in the context of two-level logic minimization using ordered binary decision diagrams of the functions and their minimal DNFs. Thus, the relationship between the number of monotone Boolean functions and the number of terms in their minimal DNF has significant practical relevance [9]. As another example, the size of the minimal DNF plays a substantial role in optimization of an important class of nonlinear digital filters, called Stack Filters [1, p. 240, 6].

In this paper, we derive asymptotic formulae for the number of monotone Boolean functions of n variables with a most probable number of terms in the minimal DNF. The results obtained here confirm the conjecture in [17] that the number of monotone Boolean functions relative to the number of terms in the minimal DNF asymptotically follows a normal distribution, with the assumption of all monotone Boolean functions being equiprobable. Fig. 1 shows such a relationship for the case of $n=7$. For the construction of this plot, the entire set of monotone Boolean functions of 7 variables had to be generated.

For establishing the asymptotic formulae, we considerably rely on results from [12] in which asymptotic formulae for $\psi(n)$ are found. Those formulae are different depending on whether n is even or odd. A similar situation exists in our case as well.

In Section 2, we give some definitions and notation as well as state the main results of this paper. Sections 3 and 4 contain the proofs of the results.

2. Notation and results

Let $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \ldots, \beta_n)$ be different n-element binary vectors. We say that $\tilde{\alpha}$ precedes $\tilde{\beta}$, denoted as $\tilde{\alpha} \prec \tilde{\beta}$, if $\alpha_i \leqslant \beta_i$ for every $1 \leqslant i \leqslant n$. Relative to the predicate $\prec$, the set of all binary vectors of a given length forms a partially ordered set.

A Boolean function $f(x_1, \ldots, x_n)$ is called *monotone* if for any two vectors $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \prec \tilde{\beta}$, we have $f(\tilde{\alpha}) \leqslant f(\tilde{\beta})$. We denote by $M(n)$ the set of all mono-

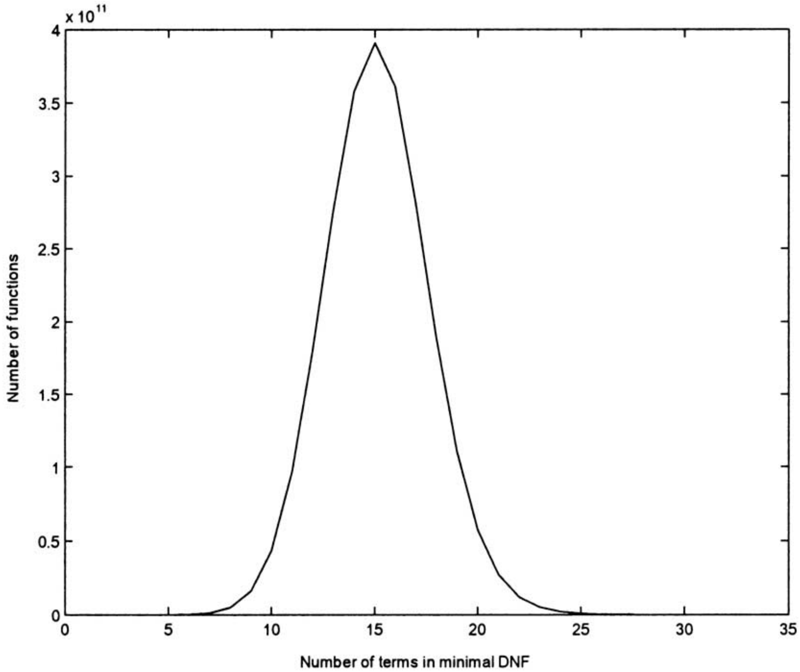

Fig. 1. Number of monotone Boolean functions of 7 variables with a given number of terms in the minimal DNF.

tone Boolean functions of n variables. Let E^n denote the n-cube, that is, a graph with 2^n vertices each of which is labeled by an n-element binary vector. Two vertices $\tilde{\alpha}=(\alpha_1,\ldots,\alpha_n)$ and $\tilde{\beta}=(\beta_1,\ldots,\beta_n)$ are connected by an edge if and only if the Hamming distance $\rho(\tilde{\alpha},\tilde{\beta})=\sum_{i=1}^{n}|\alpha_i\oplus\beta_i|=1$. The set of vectors in E^n in which there are exactly k units, $0\leqslant k\leqslant n$, is called the kth *layer* of E^n and is denoted by $E^{n,k}$.

If $f(\tilde{\alpha})=1$, then we say that $\tilde{\alpha}$ is a *true vector* of the monotone Boolean function $f(x_1,\ldots,x_n)$. A vector $\tilde{\alpha}\in E^n$ is called a *lower unit* of f if $f(\tilde{\alpha})=1$ and $f(\tilde{\beta})=0$ for any $\tilde{\beta}\prec\tilde{\alpha}$. There is a one-to-one correspondence between lower units of f and the terms in its minimal DNF. Let $\tilde{\alpha}$ be any vertex in $E^{n,k}$. Then the set of all vertices $\tilde{\beta}$ from $E^{n,k+s}$ ($s\geqslant 1$ and $k+s\leqslant n$) such that $\tilde{\beta}\succ\tilde{\alpha}$ is called the *s-shadow* of vertex $\tilde{\alpha}$. Let A be any set of vertices in $E^{n,k}$. Then the set of all vertices in $E^{n,k+s}$, each of which belongs to the s-shadow of at least one vertex in A, is called the s-shadow of the set A.

Let $A=\{\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_r\}$ be an r-element set of vertices in $E^{n,k}$. The set of vertices in A is divided into *bundles*: two vertices $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ belong to the same bundle if and only if A contains vertices $\tilde{\alpha}_{s_1},\tilde{\alpha}_{s_2},\ldots,\tilde{\alpha}_{s_k}$ such that $\rho(\tilde{\alpha}_i,\tilde{\alpha}_{s_1})=\rho(\tilde{\alpha}_{s_k},\tilde{\alpha}_j)=2$ and $\rho(\tilde{\alpha}_{s_v},\tilde{\alpha}_{s_{v+1}})=2$ for every v, $1\leqslant v\leqslant s-1$.

In [12] the following asymptotic formulas for the number of monotone Boolean functions of n variables have been found:

(a) as $n \to \infty$ over the even integers,

$$|M(n)| \sim 2^{\binom{n}{n/2}} \exp\left\{ \binom{n}{n/2-1} \left(\frac{1}{2^{n/2}} + \frac{n^2}{2^{n+5}} - \frac{n}{2^{n+4}} \right) \right\}, \tag{1}$$

(b) as $n \to \infty$ over the odd integers,

$$|M(n)| \sim 2 \cdot 2^{\binom{n}{(n-1)/2}} \exp\left\{ \binom{n}{(n-3)/2} \left(\frac{1}{2^{(n+3)/2}} - \frac{n^2}{2^{n+6}} - \frac{n}{2^{n+3}} \right) + \binom{n}{(n-1)/2} \left(\frac{1}{2^{(n+1)/2}} + \frac{n^2}{2^{n+4}} \right) \right\}. \tag{2}$$

In the process of proving formulas (1) and (2), a number of widely used properties of monotone Boolean functions were discovered. These were later used by other authors [3,15,18].

Let $M_0(n)$ denote the set of functions in $M(n)$ having the following properties. If n is even, then $M_0(n)$ contains functions $f \in M(n)$ such that all lower units of f are situated in $E^{n,n/2-1}$, $E^{n,n/2}$, and $E^{n,n/2+1}$ while for all vectors in $E^{n,n/2+2}, \dots, E^{n,n}$, function f is equal to 1. For odd n, $M_0(n)$ contains functions $f \in M(n)$ such that all lower units of f are situated in either $E^{n,(n-3)/2}$, $E^{n,(n-1)/2}$, and $E^{n,(n+1)/2}$ or $E^{n,(n-1)/2}$, $E^{n,(n+1)/2}$, and $E^{n,(n+3)/2}$. In the first case, $f(\tilde{\alpha}) = 1$ for all $\tilde{\alpha}$ in $E^{n,(n+3)/2}, \dots, E^{n,n}$ while in the second case, $f(\tilde{\alpha}) = 1$ for all $\tilde{\alpha}$ in $E^{n,(n+5)/2}, \dots, E^{n,n}$. In [12], it is proven that as $n \to \infty$, $|M(n)| \sim |M_0(n)|$. Other interesting properties of functions in $M_0(n)$ have been found in [12]. Some of them will be mentioned below.

The aim of this work is to determine asymptotic formulas for the number of functions in $M_0(n)$ with the most probable number of lower units. The results can be formulated as follows. Let us denote by $M_0(n,z)$ the set of functions $f \in M_0(n)$ such that f has z lower units.

Theorem 1. *Let n be even and $z_0 = \lfloor \frac{1}{2}\binom{n}{n/2}(1 - \frac{n/2-1}{2^{n/2}}) \rfloor$. Then, for any t, $|t| \leqslant n2^{n/2}$, as $n \to \infty$,*

$$|M_0(n, z_0 + t)| \sim \sqrt{\frac{2}{\pi\binom{n}{n/2}}}\, |M(n)| \exp\left(-2t^2 \Big/ \binom{n}{n/2} \right).$$

Theorem 2. *Let n be odd and*

$$z_0 = \left\lfloor \frac{1}{2} \binom{n}{(n-1)/2} - \binom{n}{(n-3)/2} (n-1) 2^{-(n+1)/2-3} - \binom{n}{(n-1)/2} (n-3) 2^{-(n+1)/2-2} \right\rfloor.$$

Then, for any t, $|t| \leqslant n2^{n/2}$, *as* $n \to \infty$,

$$|M_0(n, z_0 + t)| \sim \sqrt{\frac{2}{\pi\binom{n}{(n-1)/2}}}\, |M(n)| \exp\left(-2t^2 \Big/ \binom{n}{(n-1)/2}\right).$$

3. Proof of Theorem 2.1

For the case of an even n, let us suppose for the sake of brevity that

$$m = \binom{n}{n/2 - 1} = \binom{n}{n/2 + 1}. \tag{3}$$

Let us denote by $M_0(n, r, s, v, w)$ the set of functions $f \in M(n)$ such that f satisfies the following conditions:

(1) the lower units of f are only situated in the layers $E^{n,n/2-1}$, $E^{n,n/2}$, and $E^{n,n/2+1}$;
(2) the function f has r lower units in $E^{n,n/2-1}$;
(3) the set of lower units of f which are in $E^{n,n/2-1}$ splits into one-element and two-element bundles;
(4) in the set of lower units of f which are situated in $E^{n,n/2-1}$ there are s two-element bundles;
(5) the function f has v lower units in $E^{n,n/2+1}$;
(6) the set of lower units of f which are in $E^{n,n/2+1}$ splits into one-element and two-element bundles;
(7) in the set of lower units of f which are situated in $E^{n,n/2+1}$ there are w two-element bundles.

Let

$$r_0 = v_0 = \lfloor m2^{-n/2-1} \rfloor \tag{4}$$

and

$$M_0'(n) = \bigcup_{|r-r_0| \leqslant n2^{n/4}} \bigcup_{s=0}^{n^4} \bigcup_{|v-v_0| \leqslant n2^{n/4}} \bigcup_{w=0}^{n^4} M_0(n, r, s, v, w). \tag{5}$$

In [12] it is shown that, for any even n, as $n \to \infty$,

$$|M_0(n)| \sim |M_0'(n)|. \tag{6}$$

Let us denote by $M_0(n, r, s, v, w, z)$ the set of functions $f \in M_0(n, r, s, v, w)$ such that f has z lower units. If the parameters r, s, v, and w satisfy the conditions

$$|r - r_0| \leqslant n2^{n/4}, \quad |v - v_0| \leqslant n2^{n/4}, \quad 0 \leqslant s \leqslant n^4, \;\; 0 \leqslant w \leqslant n^4, \tag{7}$$

then asymptotic formulas for the number of functions $f \in M_0(n, r, s, v, w, z)$ can be obtained as follows:

(a) An r-element set A_1 containing vertices in $E^{n,n/2-1}$ is chosen. This set consists of one-element and two-element bundles and there are exactly s two-element bundles. According to Lemmas 15.4 and 15.5 [12], the number of ways to choose the set A_1 is asymptotically equal to

$$\frac{m^{r-s}}{(r-2s)!s!}\left(\frac{1}{2}\left(\frac{n^2}{4}-1\right)\right)^s \exp(-r^2n^2/8m).$$

(b) An v-element set A_2 containing vertices in $E^{n,n/2+1}$ is chosen. This set consists of one-element and two-element bundles. There are exactly w two-element bundles and none of the elements of A_2 belong to the 2-shadow of the set A_1. According to Lemmas 15.4 and 15.7 [12], the number of ways to choose the set A_2 is asymptotically equal to

$$\frac{m^{v-w}}{(v-2w)!w!}\left(\frac{1}{2}\left(\frac{n^2}{4}-1\right)\right)^w \exp\left(-\frac{v^2n^2+rvn(n+2)}{8m}\right).$$

(c) It is clear that if vertices in A_1 and A_2 are selected to be lower units, then there are $(r+v)(n/2+1)-s-w$ vertices in $E^{n,n/2}$ which cannot be used as lower units of $f \in M_0(n,r,s,v,w,z)$. In order to obtain f, it is necessary to select $z-r-v$ lower units in $E^{n,n/2}$. Thus, for a fixed A_1 and A_2, the number of ways to obtain a function $f \in M_0(n,r,s,v,w,z)$ is equal to

$$\binom{\binom{n}{n/2}-(r+v)(n/2+1)+s+w}{z-r-v}.$$

It follows from (a)–(c) that if r,s,v, and w satisfy (7), then, as $n\to\infty$,

$$\begin{aligned}|M_0(n,r,s,v,w,z)| \sim &\binom{\binom{n}{n/2}-(r+v)(n/2+1)+s+w}{z-r-v}\left(\frac{1}{2}\left(\frac{n^2}{4}-1\right)\right)^{s+w}\\ &\times \exp\left(-\frac{r^2n^2+v^2n^2+rvn(n+2)}{8m}\right)\\ &\times \frac{m^{r+v-s-w}}{(r-2s)!s!(v-2w)!w!}.\end{aligned} \tag{8}$$

In turn, if r,s,v, and w satisfy (7), then, as $n\to\infty$,

$$\frac{1}{(r-2s)!}\sim\frac{r^{2s}}{r!}\sim\frac{r_0^{2s}}{r!},\quad \frac{1}{(v-2w)!}\sim\frac{v^{2w}}{v!}\sim\frac{v_0^{2w}}{v!},$$

$$\begin{aligned}\exp\left(-\frac{r^2n^2+v^2n^2+rvn(n+2)}{8m}\right) &\sim \exp\left(-\frac{r_0^2n^2+v_0^2n^2+r_0v_0n(n+2)}{8m}\right)\\ &\sim \exp\left(-\frac{3mn^2}{2^{n+5}}-\frac{mn}{2^{n+4}}\right).\end{aligned}$$

Therefore, as $n\to\infty$,

$$|M_0(n,r,s,v,w,z)| \sim \binom{\binom{n}{n/2}-(r+v)(n/2+1)+s+w}{z-r-v}\left(\frac{1}{2}\left(\frac{n^2}{4}-1\right)\right)^{s+w} \times \exp\left(-\frac{3mn^2}{2^{n+5}}-\frac{mn}{2^{n+4}}\right)\frac{m^{r+v-s-w}r_0^{2s}v_0^{2w}}{r!s!v!w!}.$$

Furthermore, for any r,v and s,w such that $s+w>0$ and $|\frac{1}{2}\binom{n}{n/2}-z|<n2^{n/2}$, we have

$$\binom{\binom{n}{n/2}-(r+v)(n/2+1)+s+w}{z-r-v} = \binom{\binom{n}{n/2}-(r+v)(n/2+1)}{z-r-v}\prod_{i=1}^{s+w}\frac{\binom{n}{n/2}-(r+v)(n/2+1)+i}{\binom{n}{n/2}-(r+v)(n/2+1)-z+r+v+i} \sim 2^{s+w}\binom{\binom{n}{n/2}-(r+v)(n/2+1)}{z-r-v}. \tag{9}$$

Substituting (9) into (8), we see that, as $n\to\infty$,

$$|M_0(n,r,s,v,w,z)| \sim \binom{\binom{n}{n/2}-(r+v)(n/2+1)}{z-r-v}\left(\frac{n^2}{4}-1\right)^{s+w} \times \exp\left(-\frac{3mn^2}{2^{n+5}}-\frac{mn}{2^{n+4}}\right)\frac{m^{r+v-s-w}r_0^{2s}v_0^{2w}}{r!s!v!w!}. \tag{10}$$

Let $M_0'(n,z)$ denote the set of functions in $M_0'(n)$ containing z lower units. Then we have

$$|M_0'(n,z)| = \sum_{|r-r_0|\leqslant n2^{n/4}}\ \sum_{|v-v_0|\leqslant n2^{n/4}}\ \sum_{s=0}^{n^4}\sum_{w=0}^{n^4}|M_0(n,r,s,v,w,z)|. \tag{11}$$

We find the asymptotic formula for $|M_0'(n,z)|$. First, for the considered r and v, we have, as $n\to\infty$,

$$\sum_{s=0}^{n^4}\sum_{w=0}^{n^4}\left(\frac{n^2}{4}-1\right)^{s+w}\frac{r_0^{2s}v_0^{2w}}{s!w!m^{s+w}} = \sum_{s=0}^{n^4}\left(\frac{n^2}{4}-1\right)^{s}\frac{r_0^{2s}}{s!m^s}\sum_{w=0}^{n^4}\left(\frac{n^2}{4}-1\right)^{w}\frac{v_0^{2w}}{w!m^w} \sim \exp\left\{\left(\frac{n^2}{4}-1\right)r_0^2m^{-1}+\left(\frac{n^2}{4}-1\right)v_0^2m^{-1}\right\} \sim \exp\left(\frac{mn^2}{2^{n+3}}\right). \tag{12}$$

It follows from (10)–(12) that, as $n \to \infty$,

$$|M_0'(n,z)| \sim \exp\left(\frac{mn^2}{2^{n+5}} - \frac{mn}{2^{n+4}}\right)$$
$$\times \sum_{|r-r_0| \leqslant n2^{n/4}} \sum_{|v-v_0| \leqslant n2^{n/4}} \binom{\binom{n}{n/2} - (r+v)(n/2+1)}{z-r-v} \frac{m^{r+v}}{r!v!}. \tag{13}$$

Suppose that $r = r_0 + k$ and $v = v_0 + t$, where $|k| \leqslant n2^{n/4}$ and $|t| \leqslant n2^{n/4}$. Then, for $|\frac{1}{2}\binom{n}{n/2} - z| < n2^{n/2}$ and $k + t > 0$, we obtain

$$\binom{\binom{n}{n/2} - (r+v)(n/2+1)}{z-r-v} = \binom{\binom{n}{n/2} - (r_0+v_0+k+t)(n/2+1)}{z-r_0-v_0-k-t}$$
$$= \binom{\binom{n}{n/2} - (r_0+v_0)(n/2+1)}{z-r_0-v_0} \prod_{i=0}^{k+t-1} (z - r_0 - v_0 - i)$$
$$\times \frac{\prod_{i=1}^{(k+t)n/2} \{\binom{n}{n/2} - z - (r_0+v_0)n/2 - i + 1\}}{\prod_{i=1}^{(k+t)(n/2+1)} \{\binom{n}{n/2} - (r_0+v_0)(n/2+1) - i + 1\}}. \tag{14}$$

At the same time, we have, as $n \to \infty$,

$$\prod_{i=0}^{k+t-1} (z - r_0 - v_0 - i) = z^{k+t} \prod_{i=0}^{k+t-1} \left(1 - \frac{r_0+v_0+i}{z}\right)$$
$$\sim z^{k+t} \sim \left\{\frac{1}{2}\binom{n}{n/2}\right\}^{k+t}, \tag{15}$$

$$\prod_{i=1}^{(k+t)n/2} \left\{\binom{n}{n/2} - z - (r_0+v_0)n/2 - i + 1\right\}$$
$$= \left\{\binom{n}{n/2} - z\right\}^{(k+t)n/2} \prod_{i=1}^{(k+t)n/2} \left(1 - \frac{(r_0+v_0)n/2 + i - 1}{\binom{n}{n/2} - z}\right)$$
$$\sim \left\{\binom{n}{n/2} - z\right\}^{(k+t)n/2} \sim \left\{\frac{1}{2}\binom{n}{n/2}\right\}^{(k+t)n/2}, \tag{16}$$

$$\prod_{i=1}^{(k+t)(n/2+1)} \left\{\binom{n}{n/2} - (r_0+v_0)(n/2+1) - i + 1\right\}$$
$$\sim \binom{n}{n/2}^{(k+t)(n/2+1)}. \tag{17}$$

Substituting (15)–(17) into (14), we see that, as $n\to\infty$,

$$\binom{\binom{n}{n/2}-(r+v)(n/2+1)}{z-r-v}\sim\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z-r_0-v_0}2^{-(k+t)(n/2+1)}$$

$$=\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z-r_0-v_0}2^{(r_0+v_0-r-v)(n/2+1)}. \tag{18}$$

Furthermore, if $k+t<0$, then similarly, we obtain, as $n\to\infty$,

$$\binom{\binom{n}{n/2}-(r+v)(n/2+1)}{z-r-v}$$

$$=\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z-r_0-v_0}$$

$$\times\frac{\prod_{i=1}^{|k+t|(n/2+1)}\{\binom{n}{n/2}-(r_0+v_0+k+t)(n/2+1)-i+1\}}{\prod_{i=1}^{|k+t|}\{z-r_0-v_0+i\}}$$

$$\times\prod_{i=1}^{|k+t|n/2}\left\{\binom{n}{n/2}-z-(r_0+v_0)(n/2+1)+i+1\right\}$$

$$\sim\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z-r_0-v_0}2^{-(k+t)(n/2+1)}$$

$$=\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z-r_0-v_0}2^{(r_0+v_0-r-v)(n/2+1)}. \tag{19}$$

Substituting (18) and (19) into (13), we get, as $n\to\infty$,

$$|M_0'(n,z)|\sim\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z-r_0-v_0}2^{(r_0+v_0)(n/2+1)}$$

$$\times\exp\left(\frac{mn^2}{2^{n+5}}-\frac{mn}{2^{n+4}}\right)\sum_{|r-r_0|\leqslant n2^{n/4}}\sum_{|v-v_0|\leqslant n2^{n/4}}\frac{m^{r+v}}{r!v!2^{(r+v)(n/2+1)}}.$$

Since

$$\sum_{|r-r_0|\leqslant n2^{n/4}}\sum_{|v-v_0|\leqslant n2^{n/4}}\frac{m^{r+v}}{r!v!2^{(r+v)(n/2+1)}}\sim\exp\left(\frac{m}{2^{n/2}}\right),$$

we see that

$$|M_0'(n,z)| \sim \binom{\binom{n}{n/2} - (r_0+v_0)(n/2+1)}{z - r_0 - v_0} 2^{(r_0+v_0)(n/2+1)} \times \exp\left\{ m \left(\frac{1}{2^{n/2}} + \frac{n^2}{2^{n+5}} - \frac{n}{2^{n+4}} \right) \right\}. \tag{20}$$

It is well known that the binomial coefficient $\binom{n}{k}$, as a function of k, takes on the maximum value at either $k=n/2$ or $k=(n-1)/2$ depending on whether n is even or odd, respectively. In our case, let z_0 be such that $\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z-r_0-v_0}$ is maximum for $z=z_0$. It is clear that

$$z_0 = \left\lfloor \frac{1}{2}\left\{ \binom{n}{n/2} - (r_0+v_0)(n/2-1) \right\} \right\rfloor. \tag{21}$$

Using Stirling's formula (see, e.g., [5]), we see that, as $n\to\infty$,

$$\binom{\binom{n}{n/2} - (r_0+v_0)(n/2+1)}{z - r_0 - v_0} \sim \sqrt{\frac{2}{\pi\{\binom{n}{n/2} - (r_0+v_0)(n/2+1)\}}}\, 2^{\binom{n}{n/2}-(r_0+v_0)(n/2+1)} \sim \sqrt{\frac{2}{\pi\binom{n}{n/2}}}\, 2^{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}. \tag{22}$$

It follows from (20) and (22) that, as $n\to\infty$,

$$|M_0'(n,z)| \sim \sqrt{\frac{2}{\pi\binom{n}{n/2}}}\, 2^{\binom{n}{n/2}} \exp\left\{ m \left(\frac{1}{2^{n/2}} + \frac{n^2}{2^{n+5}} - \frac{n}{2^{n+4}} \right) \right\}.$$

From this as well as (1) and (3), it follows that

$$|M_0'(n,z)| \sim \sqrt{\frac{2}{\pi\binom{n}{n/2}}}\, |M(n)|. \tag{23}$$

Let $z=z_0+t$, where $|t|\leqslant n2^{n/2}$. If $t\in[0,n2^{n/2}]$, then

$$\binom{\binom{n}{n/2} - (r_0+v_0)(n/2+1)}{z - r_0 - v_0} = \binom{\binom{n}{n/2} - (r_0+v_0)(n/2+1)}{z_0 - r_0 - v_0 + t} = \binom{\binom{n}{n/2} - (r_0+v_0)(n/2+1)}{z - r_0 - v_0} \frac{\prod_{i=0}^{t-1}\{\binom{n}{n/2} - (r_0+v_0)n/2 - z_0 - i\}}{\prod_{i=1}^{t}(z_0 - r_0 - v_0 + i)}. \tag{24}$$

At the same time, using (21), we have

$$
\begin{aligned}
&\frac{\prod_{i=0}^{t-1}\{\binom{n}{n/2}-(r_0+v_0)n/2-z_0-i\}}{\prod_{i=1}^{t}(z_0-r_0-v_0+i)}\\
&=\left\{\left(\binom{n}{n/2}-(r_0+v_0)n/2-z_0\right)\Big/(z_0-r_0-v_0)\right\}^t\\
&\quad\times\frac{\prod_{i=0}^{t-1}(1-\frac{i}{\binom{n}{n/2}-(r_0+v_0)n/2-z_0})}{\prod_{i=1}^{t}(1+\frac{i}{z_0-r_0-v_0})}\\
&\sim\left\{\left(\binom{n}{n/2}-(r_0+v_0)n/2-z_0\right)\Big/(z_0-r_0-v_0)\right\}^t\\
&\quad\times\exp\left(-\frac{t^2}{2(\binom{n}{n/2}-(r_0+v_0)n/2-z_0)}-\frac{t^2}{2(z_0-r_0-v_0)}\right)\\
&\sim\left\{\left(\binom{n}{n/2}-(r_0+v_0)n/2-z_0\right)\Big/(z_0-r_0-v_0)\right\}^t\exp\left(-\frac{t^2}{z_0-r_0-v_0}\right)\\
&\sim\exp\left(-2t^2\Big/\binom{n}{n/2}\right).
\end{aligned}
\tag{25}
$$

Substituting (25) into (24), we see that, for any $t\in[0,n2^{n/2}]$,

$$
\begin{aligned}
&\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z-r_0-v_0}\\
&\sim\binom{\binom{n}{n/2}-(r_0+v_0)(n/2+1)}{z_0-r_0-v_0}\exp\left(-2t^2\Big/\binom{n}{n/2}\right).
\end{aligned}
\tag{26}
$$

Similarly, we can establish (26) if $t\in[-n2^{n/2},0]$. Substituting (26) into (20) and then using (22), (23), and (1), we see that if $|t|\leqslant n2^{n/2}$, then as $n\to\infty$,

$$|M_0'(n,z_0+t)|\sim\sqrt{\frac{2}{\pi\binom{n}{n/2}}}\,|M(n)|\exp\left(-2t^2\Big/\binom{n}{n/2}\right).$$

It remains to be shown that if $|t|<n2^{n/2}$, then

$$|M_0(n,z_0+t)\backslash M_0'(n,z_0+t)|=o(|M_0'(n,z_0+t)|).$$

This is easy to see if we use the arguments given in the proofs of Lemmas 15.1–15.3 and relationship (16.2) from [12].

This completes the proof of Theorem 2.1. □

4. Proof of Theorem 2.2

In this section the following notation will be used:

$$m=\binom{n}{(n-3)/2}=\binom{n}{(n+3)/2}, \quad p=\binom{n}{(n-1)/2}=\binom{n}{(n+1)/2}. \quad (27)$$

Let us denote by $M_{0,1}(n,r,s,v,w)$ the set of functions $f\in M(n)$ such that f satisfies the following conditions:

(1) the lower units of f are only situated in $E^{n,(n-3)/2}$, $E^{n,(n-1)/2}$, and $E^{n,(n+1)/2}$;
(2) the function f has r lower units in $E^{n,(n-3)/2}$;
(3) the set of lower units of f which are in $E^{n,(n-3)/2}$ splits into one-element and two-element bundles;
(4) in the set of lower units of f which are situated in $E^{n,(n-3)/2}$ there are s two-element bundles;
(5) the function f has v lower units in $E^{n,(n+1)/2}$;
(6) the set of lower units of f which are in $E^{n,(n+1)/2}$ splits into one-element and two-element bundles;
(7) in the set of lower units of f which are situated in $E^{n,(n+1)/2}$ there are w two-element bundles.

Further, denote by $M_{0,2}(n,r,s,v,w)$ the set of functions in $M(n)$ the definition of which differs from that of $M_{0,1}(n,r,s,v,w)$ by changing $E^{n,(n-3)/2}$ to $E^{n,(n-1)/2}$, $E^{n,(n-1)/2}$ to $E^{n,(n+1)/2}$, and $E^{n,(n+1)/2}$ to $E^{n,(n+3)/2}$. It is clear that all lower units of any function in $M_{0,2}(n,r,s,v,w)$ are located in $E^{n,(n-1)/2}$, $E^{n,(n+1)/2}$, and $E^{n,(n+3)/2}$. Let

$$r_1=\lfloor m2^{-(n+3)/2}\rfloor, \quad v_1=\lfloor p2^{-(n+1)/2}\rfloor, \quad (28)$$

$$M'_{0,1}(n)=\bigcup_{|r-r_1|\leqslant n2^{n/4}}\ \bigcup_{|v-v_1|\leqslant n2^{n/4}}\ \bigcup_{s=0}^{n^4}\bigcup_{w=0}^{n^4} M_{0,1}(n,r,s,v,w)$$

and

$$M'_{0,2}(n)=\bigcup_{|r-r_1|\leqslant n2^{n/4}}\ \bigcup_{|v-v_1|\leqslant n2^{n/4}}\ \bigcup_{s=0}^{n^4}\bigcup_{w=0}^{n^4} M_{0,2}(n,r,s,v,w).$$

In [12] it is proven that as $n\to\infty$ over the odd integers,

$$|M(n)|\sim|M'_{0,1}(n)|+|M'_{0,2}(n)|. \quad (29)$$

Denote by $M_{0,1}(n,r,s,v,w,z)$ the set of functions $f\in M_{0,1}(n,r,s,v,w)$ such that f has z lower units. If the parameters r,s,v, and w satisfy conditions

$$|r-r_1|\leqslant n2^{n/4}, \quad |v-v_1|\leqslant n2^{n/4}, \quad 0\leqslant s\leqslant n^4,\ 0\leqslant w\leqslant n^4, \quad (30)$$

then asymptotic formulas for the number of functions $f \in M_{0,1}(n,r,s,v,w,z)$ can be obtained as follows:

(a) An r-element set B_1 containing vertices in $E^{n,(n-3)/2}$ is chosen. This set consists of one-element and two-element bundles and there are exactly s two-element bundles. According to Lemmas 15.4 and 15.5 [12], the number of ways to choose the set B_1 is asymptotically equal to

$$\frac{m^{r-s}}{(r-2s)!s!}\left(\frac{1}{8}(n^2-9)\right)^s \exp(-r^2n^2/8m).$$

(b) An v-element set B_2 containing vertices in $E^{n,(n+1)/2}$ is chosen. This set consists of one-element and two-element bundles. There are exactly w two-element bundles and none of the elements of B_2 belong to the 2-shadow of the set B_1. According to Lemmas 15.4 and 15.7 [12], the number of ways to choose the set B_2 is asymptotically equal to

$$\frac{p^{v-w}}{(v-2w)!w!}\left(\frac{1}{8}(n^2-1)\right)^w \exp\left(-\frac{v^2n^2+rv(n^2+4n+3)}{8p}\right).$$

(c) It is clear that if vertices from B_1 and B_2 are selected to be lower units, then there are $r(n+3)/2+v(n+1)/2-s-w$ vertices in $E^{n,(n-1)/2}$ which cannot be used as lower units of $f \in M_{0,1}(n,r,s,v,w,z)$. In order to obtain f, it is necessary to select $z-r-v$ lower units in $E^{n,(n-1)/2}$. Thus, for a fixed B_1 and B_2, the number of ways to obtain a function $f \in M_{0,1}(n,r,s,v,w,z)$ is equal to

$$\binom{\binom{n}{(n-1)/2}-r(n+3)/2-v(n+1)/2+s+w}{z-r-v}.$$

It follows from (a)–(c) that if $r,s,v,$ and w satisfy (30), then, as $n\to\infty$,

$$\begin{aligned}|M_{0,1}(n,r,s,v,w,z)| \sim &\binom{\binom{n}{(n-1)/2}-r(n+3)/2-v(n+1)/2+s+w}{z-r-v}\\ &\times\frac{m^{r-s}p^{v-w}}{(r-2s)!s!(v-2w)!w!}\left(\frac{1}{8}(n^2-9)\right)^s\left(\frac{1}{8}(n^2-1)\right)^w\\ &\times\exp\left(-\frac{r^2n^2}{8m}-\frac{v^2n^2+rv(n^2+4n+3)}{8p}\right). \qquad (31)\end{aligned}$$

Similar to the case of an even n, we establish the following relations:

$$\frac{1}{(r-2s)!}\sim\frac{r_1^{2s}}{r!}, \quad \frac{1}{(v-2w)!}\sim\frac{v_1^{2w}}{v!}, \qquad (32)$$

$$\exp\left(-\frac{r^2n^2}{8m}-\frac{v^2n^2+rv(n^2+4n+3)}{8p}\right)$$

$$\sim \exp\left(-\frac{r_1^2 n^2}{8m} - \frac{v_1^2 n^2 + r_1 v_1 (n^2+4n+3)}{8p}\right)$$
$$\sim \exp\left(-\frac{3mn^2}{2^{n+6}} - \frac{pn^2}{2^{n+4}} - \frac{mn}{2^{n+3}}\right), \tag{33}$$

$$\binom{\binom{n}{(n-1)/2} - r(n+3)/2 - v(n+1)/2 + s + w}{z-r-v}$$
$$\sim \binom{\binom{n}{(n-1)/2} - r(n+3)/2 - v(n+1)/2}{z-r-v} 2^{s+w}. \tag{34}$$

Substituting (32)–(34) into (31), we obtain, as $n \to \infty$,

$$|M_{0,1}(n,r,s,v,w,z)| \sim \binom{\binom{n}{(n-1)/2} - r(n+3)/2 - v(n+1)/2}{z-r-v}$$
$$\times \frac{m^{r-s} p^{v-w} r_1^{2s} v_1^{2w}}{r!v!s!w!} \left(\frac{1}{4}(n^2-9)\right)^s \left(\frac{1}{4}(n^2-1)\right)^w$$
$$\times \exp\left(-\frac{3mn^2}{2^{n+6}} - \frac{pn^2}{2^{n+4}} - \frac{mn}{2^{n+3}}\right). \tag{35}$$

Denote by $M'_{0,1}(n,z)$ the set of functions $f \in M'_{0,1}(n)$ such that f has z lower units. Then we have

$$|M'_{0,1}(n,z)| = \sum_{|r-r_1| \leqslant n2^{n/4}} \sum_{|v-v_1| \leqslant n2^{n/4}} \sum_{s=0}^{n^4} \sum_{w=0}^{n^4} |M_{0,1}(n,r,s,v,w,z)|. \tag{36}$$

We find an asymptotic formula for $|M_{0,1}(n,z)|$. As in (12), we see that, as $n \to \infty$,

$$\sum_{s=0}^{n^4} \sum_{w=0}^{n^4} \left(\frac{1}{4}(n^2-9)\right)^s \left(\frac{1}{4}(n^2-1)\right)^w \frac{r_1^{2s} v_1^{2w}}{s!w!m^s p^w}$$
$$\sim \exp\left(\frac{1}{4}(n^2-9) r_1^2 m^{-1} + \frac{1}{4}(n^2-1) v_1^2 p^{-1}\right)$$
$$\sim \exp\left(\frac{mn^2}{2^{n+5}} + \frac{pn^2}{2^{n+3}}\right). \tag{37}$$

It follows from (35)–(37) that, as $n\to\infty$,

$$|M'_{0,1}(n,z)| \sim \exp\left(-\frac{mn^2}{2^{n+6}}+\frac{pn^2}{2^{n+4}}-\frac{mn}{2^{n+3}}\right) \times \sum_{|r-r_1|\leqslant n2^{n/4}} \sum_{|v-v_1|\leqslant n2^{n/4}} \binom{\binom{n}{(n-1)/2}-r(n+3)/2-v(n+1)/2}{z-r-v} \frac{m^r p^v}{r!v!}. \tag{38}$$

Let $r=r_1+k$ and $v=v_1+t$, where $|k|\leqslant n2^{n/4}$ and $|t|\leqslant n2^{n/4}$. Following the same reasoning that was used to establish (19), we obtain, as $n\to\infty$,

$$\begin{aligned}
&\binom{\binom{n}{(n-1)/2}-r(n+3)/2-v(n+1)/2}{z-r-v} \\
&\quad\sim \binom{\binom{n}{(n-1)/2}-r_1(n+3)/2-v_1(n+1)/2}{z-r_1-v_1} 2^{-k(n+3)/2-t(n+1)/2} \\
&\quad= \binom{\binom{n}{(n-1)/2}-r_1(n+3)/2-v_1(n+1)/2}{z-r_1-v_1} 2^{(r_1-r)(n+3)/2+(v_1-v)(n+1)/2}
\end{aligned} \tag{39}$$

and

$$\sum_{|r-r_1|\leqslant n2^{n/4}} \sum_{|v-v_1|\leqslant n2^{n/4}} \frac{m^r p^v}{r!v!2^{r(n+3)/2+v(n+1)/2}} \sim \exp\left(\frac{m}{2^{(n+3)/2}}+\frac{p}{2^{(n+1)/2}}\right). \tag{40}$$

Substituting (39) and (40) into (38), we have

$$|M'_{0,1}(n,z)| \sim \binom{\binom{n}{(n-1)/2}-r_1(n+3)/2-v_1(n+1)/2}{z-r_1-v_1} 2^{r_1(n+3)/2+v_1(n+1)/2} \times \exp\left(\frac{m}{2^{(n+3)/2}}+\frac{p}{2^{(n+1)/2}}-\frac{mn^2}{2^{n+6}}+\frac{pn^2}{2^{n+4}}-\frac{mn}{2^{n+3}}\right). \tag{41}$$

In this expression, the binomial coefficient, as a function of z, takes on the maximum value at

$$z-r_1-v_1=\left\lfloor \frac{1}{2}\left\{\binom{n}{(n-1)/2}-r_1(n+3)/2-v_1(n+1)/2\right\}\right\rfloor$$

that is, for

$$z=z_0=\left\lfloor \frac{1}{2}\left\{\binom{n}{(n-1)/2}-r_1(n-1)/2-v_1(n-3)/2\right\}\right\rfloor. \tag{42}$$

In this case, we have, as $n\to\infty$,

$$\binom{\binom{n}{(n-1)/2} - r_1(n+3)/2 - v_1(n+1)/2}{z_0 - r_1 - v_1}$$

$$\sim \sqrt{\frac{2}{\pi\{\binom{n}{(n-1)/2} - r_1(n+3)/2 - v_1(n+1)/2\}}}\, 2^{\binom{n}{(n-1)/2} - r_1(n+3)/2 - v_1(n+1)/2}$$

$$\sim \sqrt{\frac{2}{\pi\binom{n}{(n-1)/2}}}\, 2^{\binom{n}{(n-1)/2} - r_1(n+3)/2 - v_1(n+1)/2}. \tag{43}$$

It follows from (41) and (43) that, as $n\to\infty$,

$$|M'_{0,1}(n,z)| \sim \sqrt{\frac{2}{\pi\binom{n}{(n-1)/2}}}\, 2^{\binom{n}{(n-1)/2}}$$

$$\times \exp\left(\frac{m}{2^{(n+3)/2}} - \frac{mn^2}{2^{n+6}} - \frac{mn}{2^{n+3}} + \frac{p}{2^{(n+1)/2}} + \frac{pn^2}{2^{n+4}}\right). \tag{44}$$

Using (44), (2) and (27), we get, as $n\to\infty$,

$$|M_{0,1}(n,z)| \sim \frac{1}{2}\sqrt{\frac{2}{\pi\binom{n}{(n-1)/2}}}\,|M(n)|.$$

Repeating arguments that were used at the end of the previous section, we establish the following fact: if $n\to\infty$ and if $z=z_0+t$ such that $|t|\leqslant n2^{n/2}$, then

$$|M'_{0,1}(n,z)| \sim \frac{1}{2}\sqrt{\frac{2}{\pi\binom{n}{(n-1)/2}}}\,|M(n)|\exp\left(\frac{2t^2}{\binom{n}{(n-1)/2}}\right).$$

Since the levels $E^{n,(n-3)/2}$, $E^{n,(n-1)/2}$, and $E^{n,(n+1)/2}$ are symmetric to the levels $E^{n,(n-1)/2}$, $E^{n,(n+1)/2}$, and $E^{n,(n+3)/2}$, it follows that, for any admissible z,

$$|M'_{0,1}(n,z)| = |M'_{0,2}(n,z)|.$$

Hence, for any $z=z_0+t$, $|t|\leqslant n2^{n/2}$, and $n\to\infty$, we have

$$|M'_{0,1}(n,z)| + |M'_{0,2}(n,z)| \sim \sqrt{\frac{2}{\pi\binom{n}{(n-1)/2}}}\,|M(n)|\exp\left(\frac{2t^2}{\binom{n}{(n-1)/2}}\right).$$

In order to conclude the proof of this theorem, it remains to substitute the values of parameters r_1 and v_1 into (42) and use (28) and (27) as well as to show that if $|t|<n2^{n/2}$, then

$$|(M_{0,1}(n,z_0+t)\cup M_{0,2}(n,z_0+t))\backslash(M'_{0,1}(n,z_0+t)\cup M'_{0,2}(n,z_0+t))|$$

$$= \mathrm{o}(|M'_{0,1}(n,z_0+t)|).$$

This is easy to see if we use the arguments given in the proofs of Lemmas 15.1–15.3 and relationship (17.2) from [12].

This proves Theorem 2.2. □

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